Forum for Electromagnetic Research Methods and Application Technologies (FERMAT)

Fast Direct Volume Integral Equation Solvers For Large-Scale General Electromagnetic Analysis by Saad Omar

Final Examination (April 23, 2014)
School of Electrical and Computer Engineering Purdue University

Abstract: Among existing computational electromagnetic methods, volume integral equation (VIE) based methods have unique advantages in modeling both open-region problems and complicated geometry and materials. However, to accentuate the unique advantages of the VIE-based methods, two major obstacles must be overcome: one is the generality of the VIE formulation; the other being the high computational cost of a VIE-solver.

Traditional VIE-based formulations developed for solving wave-related problems are not amenable for solving circuit problems, while existing circuit-based VIE-formulations involve simplifications and approximations that are invalid for wave-related problems. In this work, we develop a new first-principles-based VIE-formulation that bridges the gap between wave- and circuit-based electromagnetic analysis, using which the analysis and design of circuits exposed to external electromagnetic fields is made possible in a full electromagnetic spectrum.

The linear system of equations resulting from a VIE-based analysis is not only dense but also large involving volume unknowns in a 3-D computational domain. To address this computational challenge, we overcame the related numerical issues to develop an H2-matrix based linear complexity direct VIE solver for large-scale circuit parameter extraction, which is capable of solving millions of VIE unknowns using modest computational resources on a single CPU core. Lastly, our newly developed minuscule cost SVD-mimicking 2-matrix recompression schemes have made it possible, for the first time, to achieve linear complexity iterative and fast direct solvers for general large-scale electrodynamic scattering problems.

Keywords: Fast Direct Inverse, Linear complexity direct VIE solvers, Volume Integral Equation, Direct Circuit Solver, Fast electrodynamic solvers, Simultaneous circuit radiation formulations.

This use of this work is restricted solely for academic purposes. The author of this work owns the copyright and no reproduction in any form is permitted without written permission by the author.*

Outline

- Motivation And Challenges
- Novel First Principles VIE Formulations
- Fast VIE Solvers For Large-Scale EM Analysis
  - *O*(*N*) Direct VIE Full-wave Circuit Solver
  - *O*(*N*) Iterative VIE Electrodynamic Solver
  - *O*(*NlogN*) Direct VIE Electrodynamic Solver
- Summary Of Accomplishments
- Future Research Vectors

Motivation

- **Open region scattering**
  - Scattering analysis (medical diagnostics, military applications)
- **Circuits with inhomogeneous materials**
  - VLSI circuits (multiple dielectrics, arbitrarily shaped lossy conductors)
- **Simultaneous open-region scattering and circuit analysis**
  - Sensitive microwave, VLSI and RF circuits
  - Severe ambient conditions
  - Communication satellite and military circuits

Motivation and Challenges

- Computational EM methods:
  - PDE based (FEM and FDM)
    - Modeling inhomogeneous materials and irregular structures
    - Absorbing boundary condition for open region analysis
      - Approximate
      - Introduce additional unknowns and computation
  - IE based (surface(SIE) and volume(VIE))
    - Analytical open region modeling: exact, avoid additional computation
    - Modeling homogeneous(SIE) and inhomogeneous(VIE) problems
    - For same N, computationally expensive than PDE
  - For simultaneous circuit-scattering analysis
    - Method for choice — VIE

Motivation and Challenges

- **Challenges:**
  - **Missing First-Principles Based VIE Formulation for Simultaneous Circuit-Scattering Analysis**
    - **Approx. VIE circuit analysis not amenable for wave analysis**
    - **Current wave-based VIE not applicable to circuit extraction**
  - **Computational Burden**
    - ***Cubic* growth in number of unknowns**
    - **Poor numerical conditioning of circuit regime EM problems[1]**
    - **Large (off-diagonal) rank for electrodynamic problems[2]**
    - **Rank growth with electric size of the problem**

[1] J. Zhu, S. Omar, W. Chai and D. Jiao, "A rigorous solution to the low frequency breakdown in the electric field integral equation," *2011 IEEE International Symposium on Antennas and Propagation (APSURSI)*, vol., no., pp.3214,3217, July 2011.

[2] W.Chai and D. Jiao, " A theoretical study on the rank of integral operators for broadband electromagnetic modeling from static to electrodynamic frequencies," *IEEE Trans. on Components, Packaging and Manufacturing Tech.*, Dec. 2013

PURDUE
UNIVERSITY

Novel First-Principles Based VIE Formulation

Current State-of-the-art Methods

- Current VIE Circuit Extractors
  - State-of-the-art Methods
    - PEEC-based[1]

      Involve circuit-based simplifications and approximations

    - Fast Henry[2]

      Quasi-static analysis for straight conductors
  - Approximations not amenable for wave-based analysis

[1] A. E. Ruehli, “Equivalent Circuit Models for Three Dimensional Multi conductor Systems,” IEEE Trans. MTT, Mar. 1974.
[2] FastHenry- http://www.rle.mit.edu/cpg/research_codes.htm

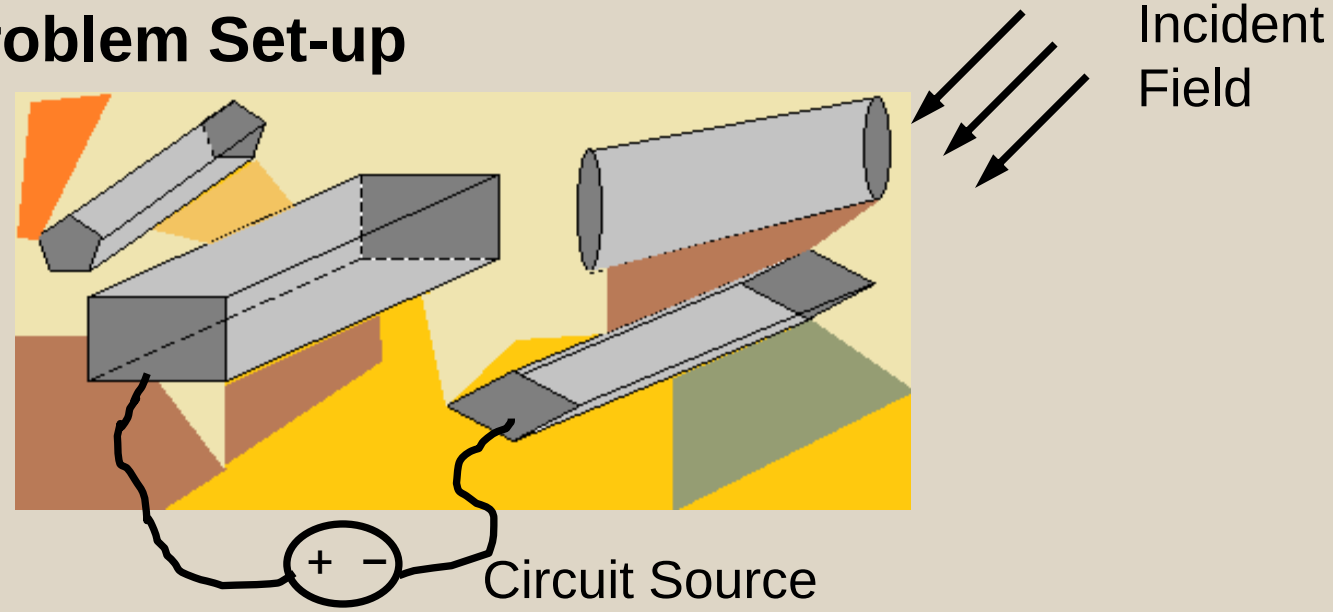
First-Principles Based Formulation

- **VIE solvers for wave problems offer**
  - **Theoretical rigor** and validity in **full electromagnetic spectrum**
  - **Tetrahedral-element discretization** gives flexibilities in modeling arbitrarily shaped conductors and dielectrics
  - **SWG vector basis functions** capable of capturing currents flowing along an arbitrary direction

☞ **Incident field and voltage gap excitation used in a traditional VIE *wave problem* do NOT facilitate *circuit parameter extraction* with ports arbitrarily located in the physical layout of a circuit.**

Formulation Challenges

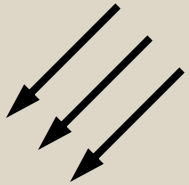
General Problem Set-up



- Known wave-based VIE deals perfectly with incident field excitation
- How to model the circuit source without losing benefits of VIE?
- New modeling formulation should be full-wave and first principles so that superposition can finally be applied for simultaneous analysis

First-Principles Based Formulation

- **Known Incident Field Based Excitation**



System of Equations to Solve

$$\vec{E}(\vec{r}) = \vec{E}^{s}(\vec{r}) + \vec{E}^{i}(\vec{r})$$

$$\vec{E}(\vec{r}) + j\omega\vec{A}^{s}(\vec{r}) + \nabla\phi^{s}(\vec{r}) = \vec{E}^{i}(\vec{r})$$

$$\vec{D}(\vec{r})/\bar{\varepsilon}(\vec{r}) - \omega^{2}\iiint_{V}\mu\kappa(\vec{r}\,')\vec{D}(\vec{r}\,')G_{0}(\vec{r},\vec{r}\,')dv' + \nabla\phi^{s} = \vec{E}^{i}(\vec{r})$$

where:

$$\phi(\vec{r}) = -\iiint_{V}\nabla'\cdot(\kappa(\vec{r}\,')\vec{D}(\vec{r}\,')/\varepsilon_{0})\,G_{0}(\vec{r},\vec{r}\,')dv'$$

$$= -\iiint_{V}\kappa\nabla'\cdot\vec{D}(\vec{r}\,')/\varepsilon_{0}\,G_{0}(\vec{r},\vec{r}\,')dv' + \iint_{S}\vec{D}\cdot\hat{n}(\kappa^{+}-\kappa^{-})/\varepsilon_{0}\,G_{0}(\vec{r},\vec{r}\,')ds'$$

$$\kappa(\vec{r}\,') \equiv \text{contrast ratio} = \left(\bar{\varepsilon}(\vec{r}\,') - \varepsilon_{0}\right)/\bar{\varepsilon}(\vec{r}\,')$$

First-Principles Based Formulation

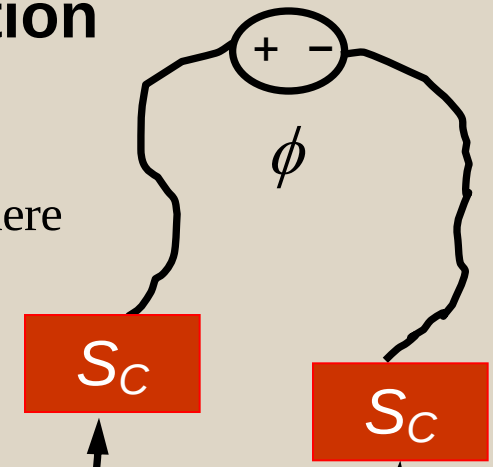
❑ Proposed Potential Based Circuit Extraction

System of Equations to Solve

$$\vec{E}(\vec{r}) = -\nabla\phi - j\omega\vec{A}(\vec{r})$$

subject to $\phi(\vec{r}_c) = \psi_C$

$\vec{r}_c$ denotes a point where ψ_C is applied



$$\vec{D}(\vec{r})/\bar{\varepsilon}(\vec{r}) - \omega^2 \iiint_V \mu\kappa(\vec{r}')\vec{D}(\vec{r}')G_0(\vec{r},\vec{r}')dv' + \nabla\phi = 0$$

subject to $\phi(\vec{r}_c) = \psi_C$

$$\phi(\vec{r}) = \iiint_V \frac{-\kappa(\vec{r}')\nabla'\cdot\vec{D}(\vec{r}')}{\varepsilon_0} G_0(\vec{r},\vec{r}')dv' + \iint_{S\notin S_C} \frac{(\kappa^+ - \kappa^-)\vec{D}\cdot\hat{n}}{\varepsilon_0} G_0(\vec{r},\vec{r}')ds' + \iint_{S_C} \frac{\rho_S}{\varepsilon_0} G_0(\vec{r},\vec{r}')ds'$$

Additional unknowns

First-Principles Based Formulation

❑ Proposed Potential Based Circuit Extraction

- The unknowns to solve

$$\left(\vec{D}\,,\,\rho_S\Big|_{S\in S_C}\right)$$

- After **D** is solved, compute

$$I = j\omega\iint_{X\,\mathrm{sec}} dsD_n$$

$$Z = V / I$$

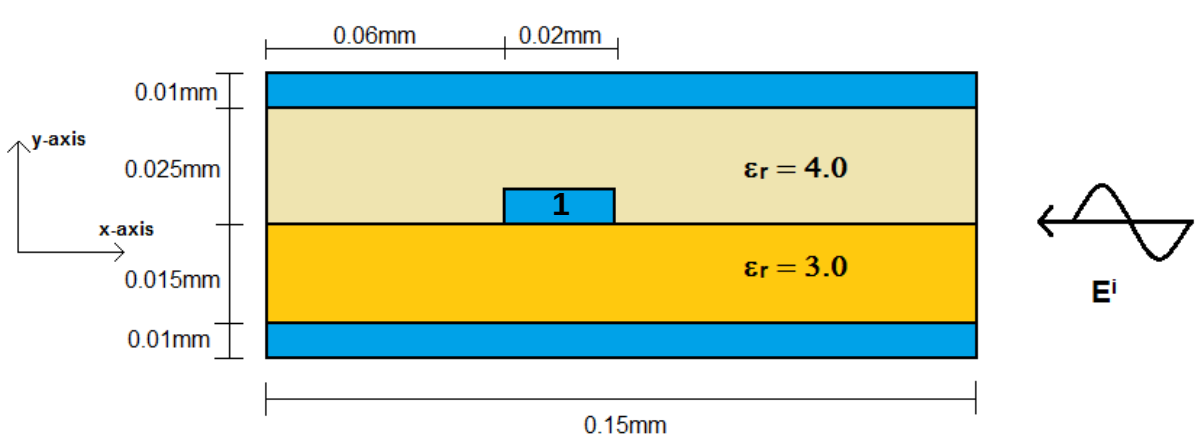
- Obtain admittance (**Y**-), scattering (**S**-) parameters, etc.

Intel Package Interconnect In External Fields

Frequency Range: 1-30 GHz Plane Thickness: 0.01 mm

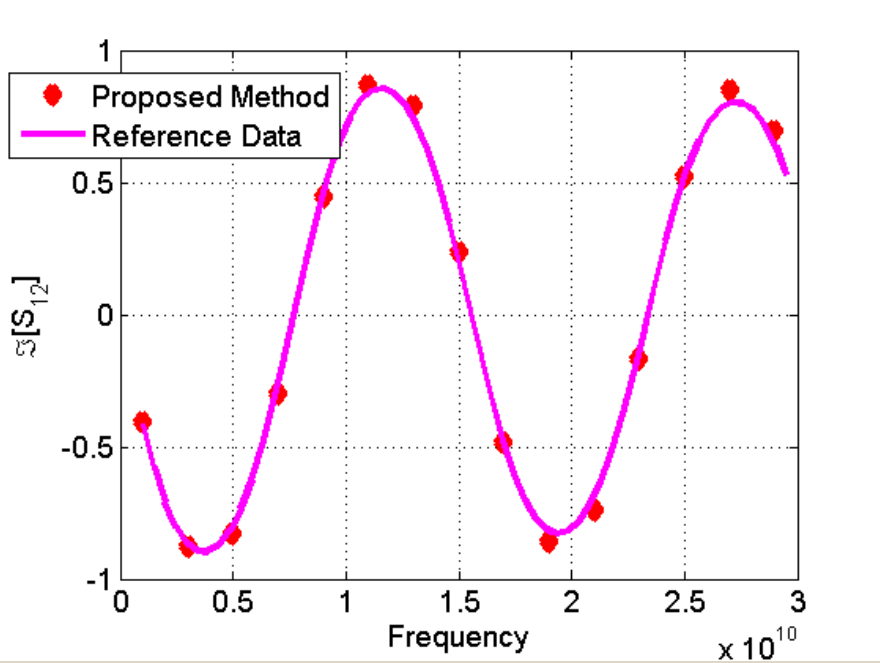
External Incident Field:

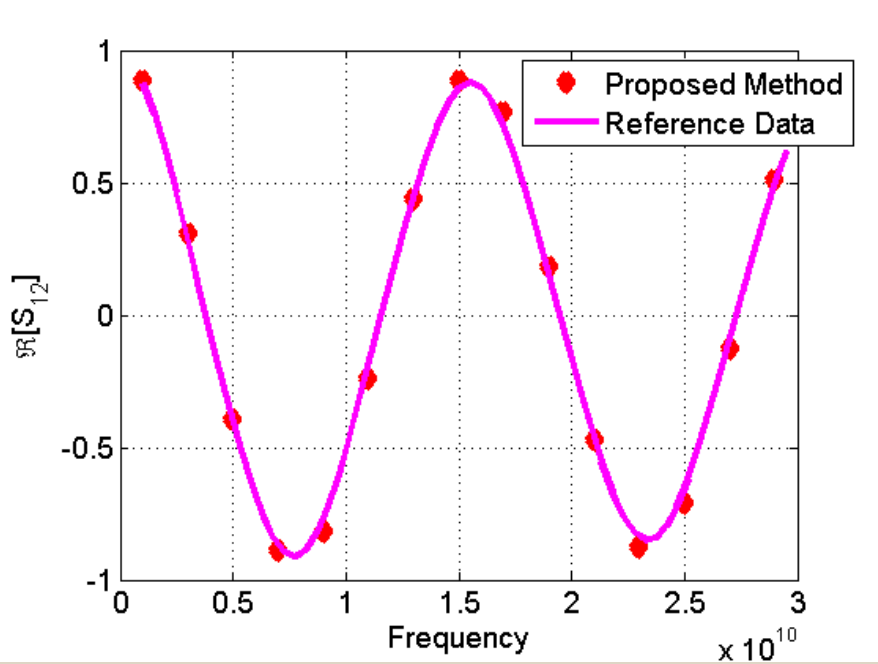
- Any freq., polarization and direction can be simulated
- a_y polarized, $-a_x$ directed plane wave
- Frequency range same as circuit source



Cross-sectional view (length = 1cm)

Intel Package Interconnect

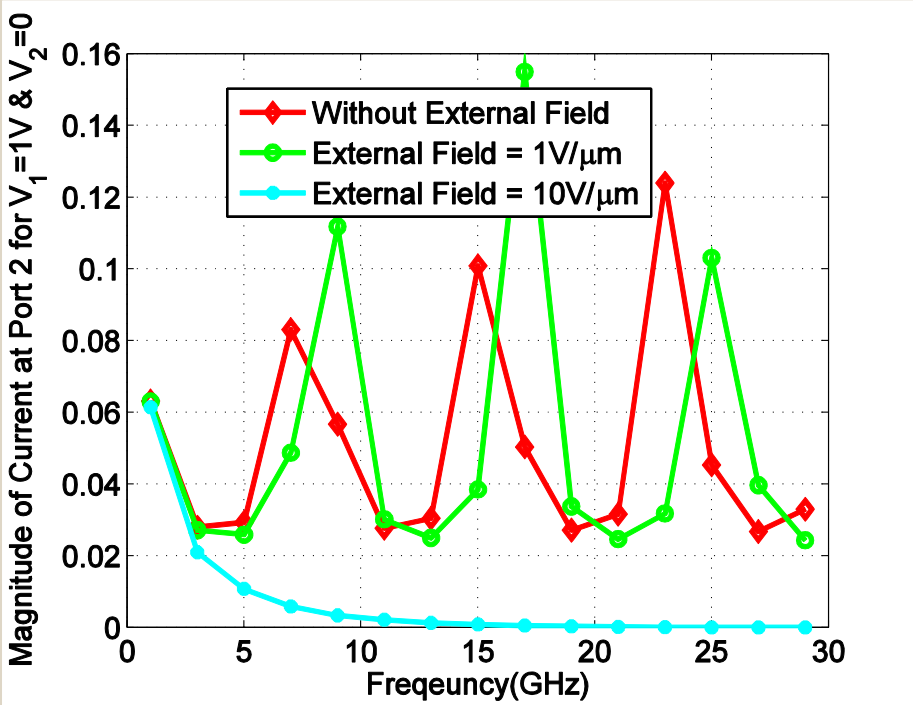


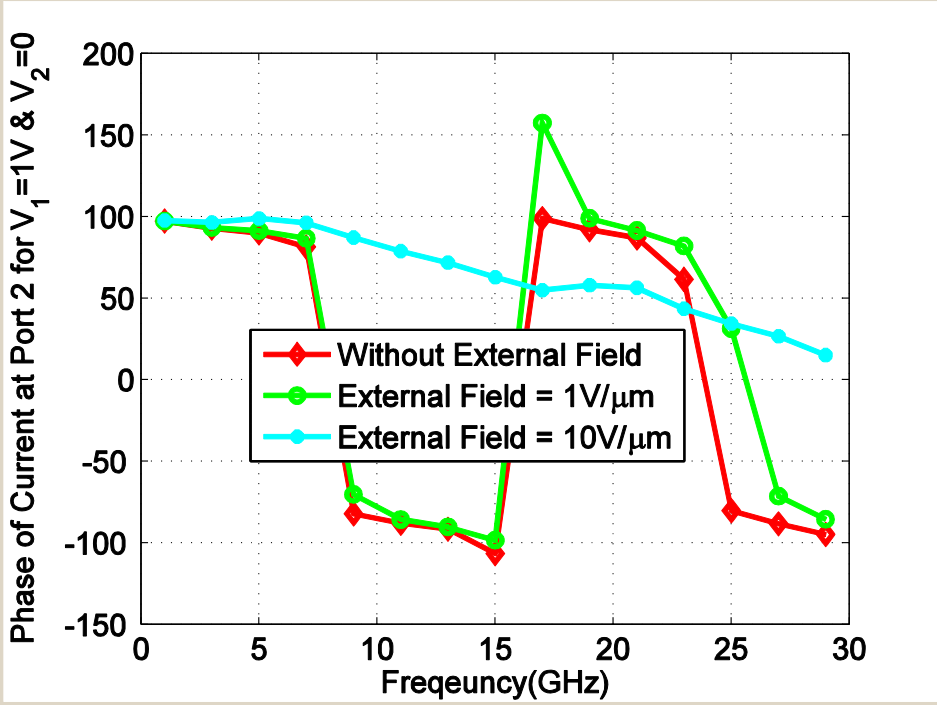


S_{12} Real and Imaginary Parts

Courtesy of Intel

Intel Package Interconnect In External Fields





Circuit Parameters in the Presence of External Fields

Fast Solvers For Large-Scale Electromagnetic Analysis

Current State-of-the-art Methods

- Current Fast IE solvers
  - Iterative Solvers
    - At best $N_{rhs}N_{it}O(NlogN)$
    - Application restricted by *iteration count* and *no. of rhs*
    - Examples: FFT- , FMM- and AIM- based solvers
  - Direct Solvers
    - Surface IE direct solvers e.g.
    - LOGOS-based : $O(N^{1.8})$ for 2-D numerically
    - ACA-based : $O(N^2)$ numerically
    - Both lack theoretical complexity bound

$\mathcal{H}^2$-matrix State of the Art

- In mathematical literature

  For frequency independent kernels

  - *O*(*N*) storage and matrix-vector multiplication
  - *O*(*N*) matrix-matrix multiplication
  - No *O*(*N*) complexity established for direct inverse
  - No *O*(*N*) complexity established for LU factorization

- Recent work in our lab

  For frequency independent and electrically moderate problems

  - *O*(*N*) complexity established for SIE direct inverse and LU factorization [1,2]
  - No *O(N)* direct VIE inverse

  For electrically large problems

  - A theoretical proof on the error bounded low-rank representation [3]
  - Fast $\mathcal{H}$ and $\mathcal{H}^2$ based iterative and direct SIE solvers [4-5]
  - No *O(N)* iterative or *O(NlogN)* direct electrodynamic solver developed for any IE

[1-3] Chai and Jiao, *IEEE Trans. MTT*, 2013; *IEEE Trans. Adv. Packaging*, 2012; *IEEE Trans. CPMT*, 2013
[4-5] Chai and Jiao, ACES 2012; *IEEE Trans. AP*, 2009

$\mathcal{H}^2$-matrix State of the Art

❑ $\mathcal{H}^2$-matrix Definition

Hierarchical low-rank representation

In an $\mathcal{H}^2$-matrix **C**, all the off-diagonal blocks $\mathbf{C}_{m\times n}$ that dictate the interaction between two geometrically separated blocks can be written as

$$\mathbf{C}_{m\times n} = \mathbf{V}_{m\times k}\mathbf{S}_{k\times k}\mathbf{W}_{n\times k}^{\mathsf{T}}$$

where (1) $k < \min(m, n)$, and hence being low rank

(2) Cluster basis **V** and **W** are nested

[1] W. Hackbusch, B. Khoromskij, and S. Sauter, "On $\mathcal{H}^2$–matrices," Lecture on Applied Mathematics, H. Bun-gartz, R. Hoppe, and C. Zenger, eds., pp. 9-29, 2000.

[2] S. Börm, "$\mathcal{H}^2$-matrix arithmetics in linear complexity," *Computing*, 77: 1-28, 2006.

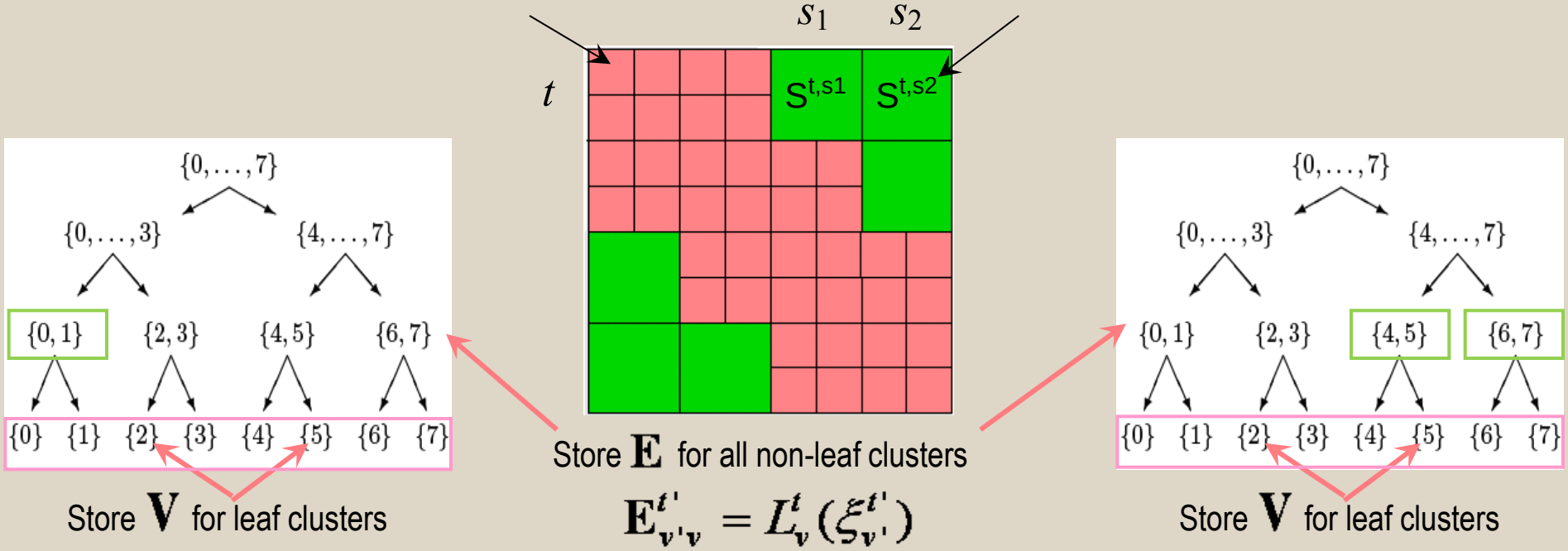
$\mathcal{H}^2$-matrix Representation

Nested Property

$$\mathbf{V}^t = \begin{pmatrix} \mathbf{V}^{t1}\mathbf{E}^{t1} \\ \mathbf{V}^{t2}\mathbf{E}^{t2} \end{pmatrix} = \begin{pmatrix} \mathbf{V}^{t1} & \\ & \mathbf{V}^{t2} \end{pmatrix} \begin{pmatrix} \mathbf{E}^{t1} \\ \mathbf{E}^{t2} \end{pmatrix}$$

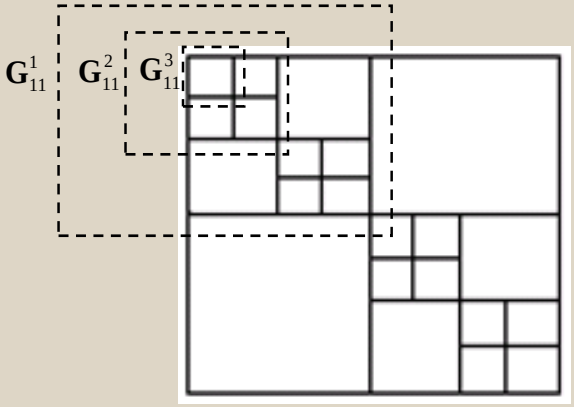
$\mathbf{G}^{t,s}$ **Store full matrix**

$\tilde{\mathbf{G}}^{t,s} := \mathbf{V}^t \mathbf{S}^{t,s} \mathbf{V}^{sT}$ **Store S only**



$\mathcal{H}^2$-matrix Recursive Inverse

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix}$$



Recursive inversion

$$\mathbf{G}^{-1} = \begin{bmatrix} \mathbf{G}_{11}^{-1} \oplus \mathbf{G}_{11}^{-1} \otimes \mathbf{G}_{12} \otimes \mathbf{S}^{-1} \otimes \mathbf{G}_{21} \otimes \mathbf{G}_{11}^{-1} & -\mathbf{G}_{11}^{-1} \otimes \mathbf{G}_{12} \otimes \mathbf{S}^{-1} \\ -\mathbf{S}^{-1} \otimes \mathbf{G}_{21} \otimes \mathbf{G}_{11}^{-1} & \mathbf{S}^{-1} \end{bmatrix}$$

where, $\mathbf{S} = \mathbf{G}_{22} \oplus (-\mathbf{G}_{21} \otimes \mathbf{G}_{11}^{-1} \otimes \mathbf{G}_{12})$

$O(k_1^3)$

Complexity Analysis

- Memory Requirements

$$St(\mathcal{H}^2\text{-matrix}) = St(\text{leaf clusters}) + St(\text{nonleaf clusters}) + St(\text{admissible blocks}) + St(\text{inadmissible blocks})$$

$$= St(\mathbf{V}^t) + St(\text{transfer matrix } \mathbf{E}^{t'}) + St(\text{coupling matrix } \mathbf{S}^b) + St(\text{full matrix } \mathbf{G}^b)$$

$$= \sum_{t\in\mathcal{L}_\mathcal{I}} O(k_1(t))\#\hat{t} + \sum_{t\in T_\mathcal{I}\setminus\mathcal{L}_\mathcal{I}} \sum_{t'\in sons(t)} O(k_1(t)k_1(t')) + \sum_{b=(t,s)\in\mathcal{L}^+_{\mathcal{I}\times\mathcal{I}}} O(k_1(t)k_1(s)) + \sum_{b=(t,s)\in\mathcal{L}^-_{\mathcal{I}\times\mathcal{I}}} \#\hat{t}\,\#\hat{s}$$

$$\le O(k_1(t))\cdot N + 2\sum_{t\in T_\mathcal{I}} O(k_1{}^2(t)) + C_{sp}\sum_{t\in T_\mathcal{I}} O(k_1{}^2(t)) + C_{sp}\sum_{t\in T_\mathcal{I}} n_{\min}^2$$

$$\le O(k_1)\cdot N + 2O(k_1{}^2)N + O(C_{sp}k_1{}^2)N + 2C_{sp}n_{\min}^2 N = O(N)$$

- Inversion Time

$$Comp(\mathbf{G}^{-1}) = \sum_{l=0}^{L}(\#\text{blocks at level } l)O(C_{sp}k_1^3) \le \sum_{l=0}^{L} 2^l C_{sp}O(C_{sp}k_1^3)$$

$$= C_{sp}O(C_{sp}k_1^3)\#T_\mathcal{I} \le O(C_{sp}^2 k_1^3)N$$

O(*N*) Direct VIE Full-wave Circuit Solver

Computational Challenges

- Application of $\mathcal{H}^2$-inverse to Circuit EM problems

  - All possible shapes and types of system matrices

  Proposed Solution: Elimination to get conformal matrices

  - Conditioning of the VIE system becomes pivotal

  - Interpolation based rank is not optimal

Computational Challenges

- Application of $\mathcal{H}^2$-inverse to Circuit EM problems

  - All possible shapes and types of system matrices

  Proposed Solution: Elimination to get conformal matrices

  - **Conditioning of the VIE system becomes pivotal…**

Well Conditioned First-Principles Formulations

1. Avoid addition of magnetic and electric potential matrices[1]
2. Explicitly enforce physical condition $\nabla \cdot D = 0$

$$\vec{E}(\vec{r}) = -\nabla\phi - j\omega\vec{A}(\vec{r})$$

$$\text{subject to } \phi(\vec{r}_c) = \psi_C$$

$$\nabla \cdot D(\vec{r}) = 0, \quad \vec{r} \in V$$

Caution:

Order of elimination is also critical to satisfy condition (1)

[1] J. Zhu, S. Omar, W. Chai and D. Jiao, “A rigorous solution to the low frequency breakdown in the electric field integral equation,” *2011 IEEE International Symposium on Antennas and Propagation (APSURSI)*, vol., no., pp.3214,3217, July 2011.

Well Conditioned First-Principles Formulations

1. Avoid addition of magnetic and electric potential matrices[1]
2. Explicitly enforce physical condition $\nabla \cdot D = 0$

$$\vec{D}(\vec{r})/\bar{\varepsilon} - \omega^2 \iiint_V \mu\kappa(\vec{r}\,')\vec{D}(\vec{r}\,')G_0(\vec{r},\vec{r}\,')dv' + \nabla\phi = 0$$

$$\text{subject to } \phi(\vec{r}_c) = \psi_C$$

$$\nabla \cdot D(\vec{r}) = 0, \quad \vec{r} \in V$$

Caution:

Order of elimination is also critical to satisfy condition (1)

[1] J. Zhu, S. Omar, W. Chai and D. Jiao, “A rigorous solution to the low frequency breakdown in the electric field integral equation,” *2011 IEEE International Symposium on Antennas and Propagation (APSURSI)*, vol., no., pp.3214,3217, July 2011.

Well Conditioned Formulations

$$\begin{bmatrix} \mathbf{Z} & \mathbf{V^{t1}} & \mathbf{V^{t2}} \\ \boldsymbol{\Sigma} & 0 & 0 \\ -\mathbf{P}_0^{OB} & 0 & \boldsymbol{I} \end{bmatrix} \begin{Bmatrix} D_n \\ \phi_V \\ \phi_{OB} \end{Bmatrix} = \begin{Bmatrix} K_1 \\ 0 \\ K_2 \end{Bmatrix}$$

$$(\mathbf{Z})_{mn} \equiv \int_V \vec{f}_m(\vec{r}).\vec{f}_m(\vec{r})/\varepsilon(\vec{r})dv - (\omega^2\mu)\int_V dv\kappa\,\vec{f}_m(\vec{r})\int_{V'} dv'\,\vec{f}_n(\vec{r}\,')G_0$$

$$\mathbf{P}_0 = (1/\varepsilon_0)\iint_S G_0(\vec{r},\vec{r}\,')\,ds' \qquad \mathbf{P}_0^{OB} = \mathbf{P}_{0OB,disc} - \mathbf{P}_{0OB,C}\mathbf{P}_{0C,C}^{-1}\mathbf{P}_{0OB,disc}$$

$$(\boldsymbol{\Sigma})_{mn} \equiv (Sparse)\nabla.f_n \ for\ each\ of\ m^{th}\ tetrahedron's\ patch$$

$$(\mathbf{V^{t1}})_{m*} \equiv (Sparse)Coupling\ of\ m^{th}\ basis\ with\ its\ tetrahedron$$

$$(\mathbf{V^{t2}})_{m*} \equiv (Sparse)Coupling\ of\ m^{th}\ basis\ with\ its\ outer\ boundary(OB)\ patch$$

$$(K_1)_m \equiv Coupling\ (vector)\ of\ m^{th}\ basis\ with\ its\ Contact\ patch$$

$$(K_2)_m \equiv Contribution\ of\ \mathbf{P}_0\ from\ Contact\ patches = \mathbf{P}_{0OB,C}\mathbf{P}_{0C,C}^{-1}\phi_C$$

$$(\mathbf{P}_{0I,J})_{mn} \equiv (Dense)Potential\ at\ I-type\ patches\ due\ to\ J-type\ charges$$

Well Conditioned Formulations

- Step1: Eliminate ϕ_V and D_n to get ϕ_{OB}

$$\phi_{OB} = [\mathbf{I} + \mathbf{F}\mathbf{V}^{\mathbf{t2}}]^{-1}(K_2 + \mathbf{F}K_1)$$

$$\mathbf{F} = \mathbf{P}_0^{OB}\mathbf{Z}^{-1}[\mathbf{I} - \mathbf{V}^{\mathbf{t1}}\mathbf{T}]^{-1}$$

$$\mathbf{T} = [\mathbf{\Sigma}\mathbf{Z}^{-1}\mathbf{V}^{\mathbf{t1}}]^{-1}\mathbf{\Sigma}\mathbf{Z}^{-1}$$

- Step2: Now get ϕ_V from D_n eliminated expression

$$\phi_V = \mathbf{T}(K_1 - \mathbf{V}^{\mathbf{t2}}\phi_{OB})$$

- Step3: Finally get D_n

$$D_n = \mathbf{Z}^{-1}(K_1 - \mathbf{V}^{\mathbf{t1}}\phi_V - \mathbf{V}^{\mathbf{t2}}\phi_{OB})$$

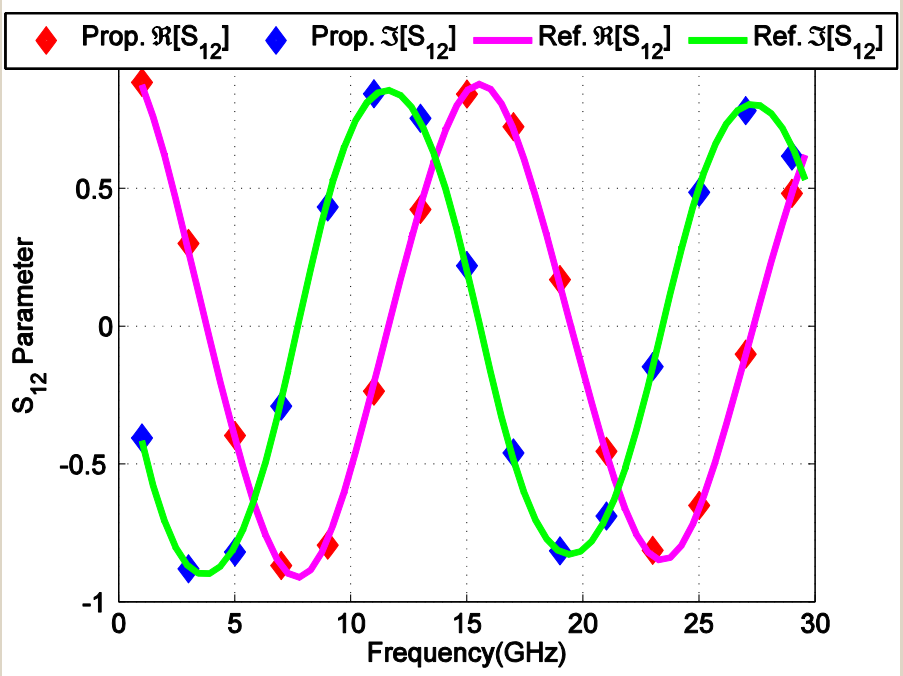
Case 1: Multi-Dielectric Package Interconnect

Conductivity: 5.8×10^7 S/m

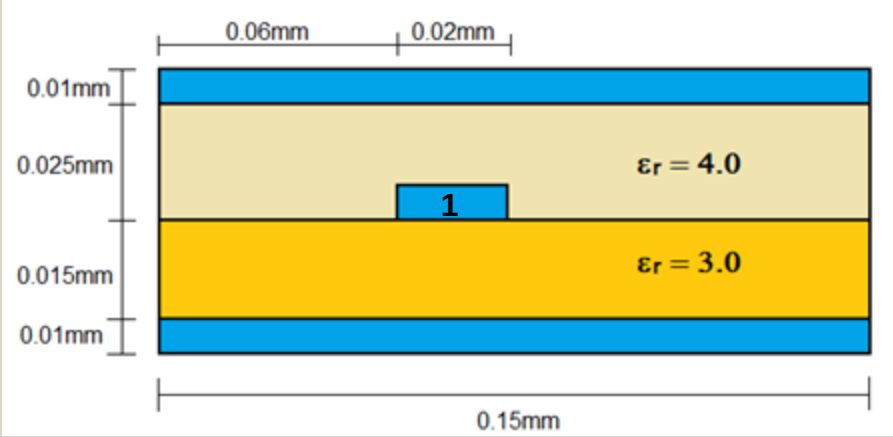
Frequency Range: 1-30 GHz

Metal Thickness: 0.01 mm

Plane Thickness: 0.01 mm



Cross-sectional view



Length : 1 cm

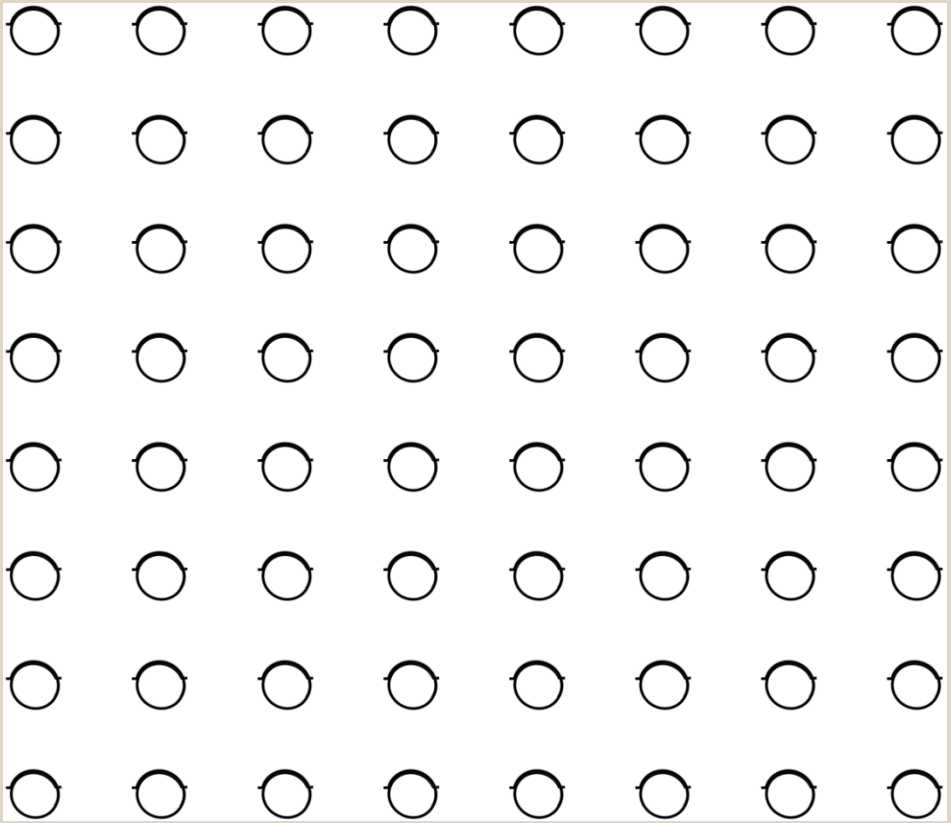
Courtesy of Intel

Case 2: Large Scale Array of Ring Inductors

Inductors: 2x2 to 64x64
No. of interpolation pt.'s: 1
Frequency point: 10 GHz

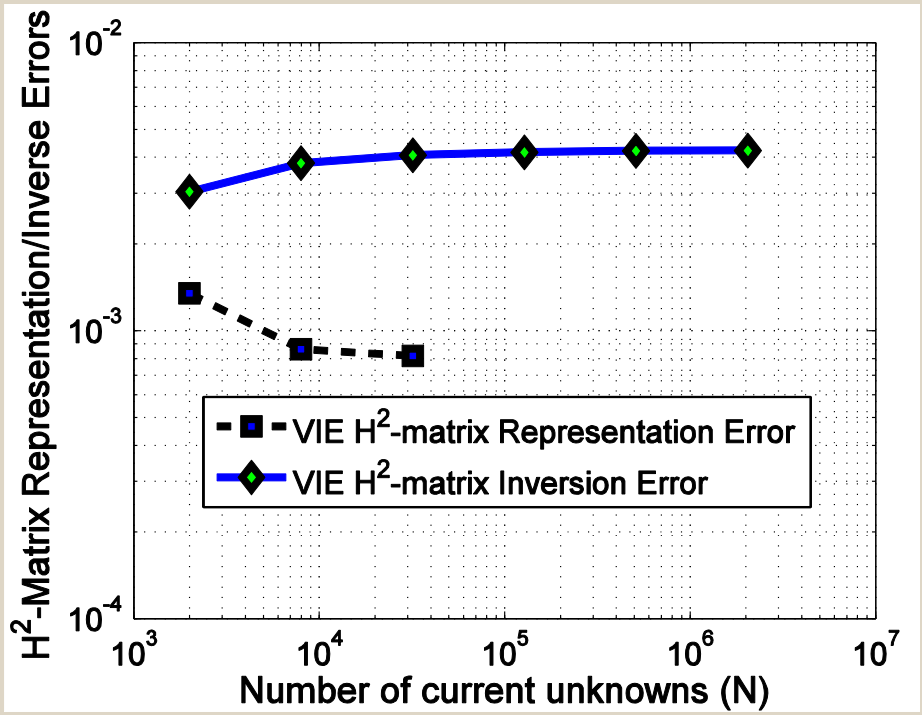
Number of Current Unknowns:
1992 to 2,039,808

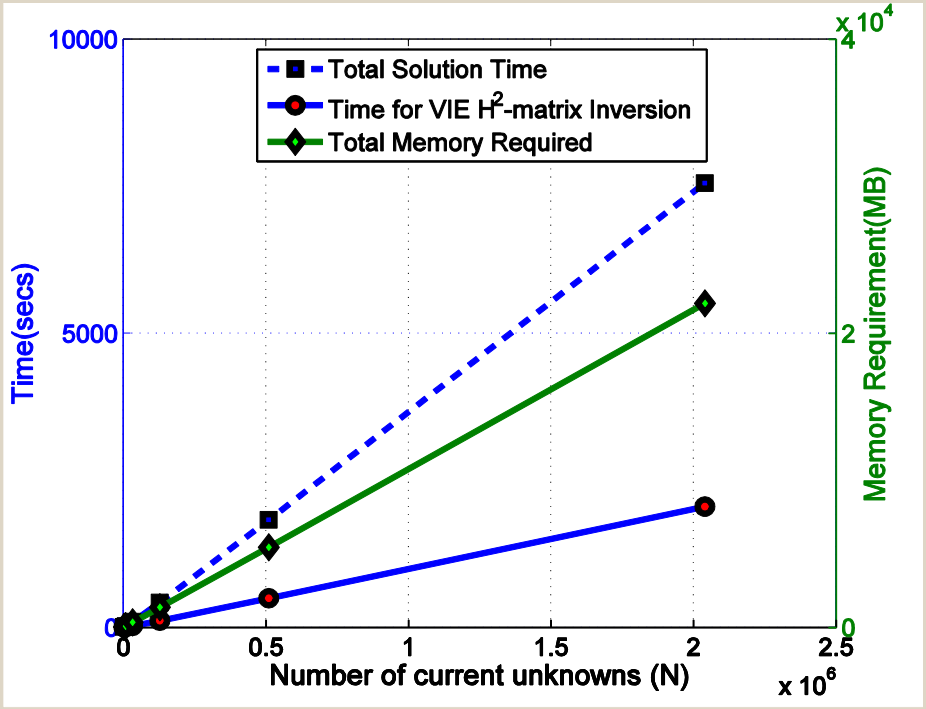
Total number of Unknowns:
3744 to 3,833,856



An 8 x 8 Ring Inductor Array

Case 2: Large Scale Array of Ring Inductors





Performance Plots for large-scale Inductor array example

Computational Challenges

- Application of $\mathcal{H}^2$-inverse to Circuit EM problems
  - All possible shapes and types of system matrices

  Proposed Solution: Elimination to get conformal matrices
  - Conditioning of the VIE system becomes pivotal
  - **Interpolation based rank is not optimal …**

Improved Rank-Minimization Procedure

For Leaf Clusters (t)

- Step1:

$$G_2^t = V^t[\mathbf{S}^{t_{sum}}_{self} + E^t \mathbf{S}^{t_{sum}}_{ancestors} E^{tH}]V^{tH} + \left(\sum_{s \in col^-(t)} F^{(t,s)} F^{(t,s)H}\right) \quad [1]$$

Full-rank contribution
thus sub-optimal

- Step2:

$$G_2^t \underset{O(k^3)}{\overset{SVD}{=}} PDP^H$$

- Step3:

$$\tilde{V}^t = P \qquad \tilde{B}^t = \tilde{V}^{tH} V^t$$

[1] W. Chai and D. Jiao, “Linear complexity direct and iterative integral equation solvers accelerated by a new rank-minimized $\mathcal{H}^2$–representation for large-scale 3-D interconnect extraction,” IEEE Trans. on MTT, vol. 61, no. 8, pp. 2792-2805, 2013.

Improved Rank-Minimization Procedure

For Non-Leaf Clusters (t) (same as in [1])

- Step1:

$$G_{2,\text{proj}}^{t} = \begin{bmatrix} \tilde{B}^{t1}E^{t1} \\ \tilde{B}^{t2}E^{t2} \end{bmatrix} \left(\mathbf{S}_{\text{self}}^{t_{\text{sum}}} + E^{t}\mathbf{S}_{\text{ancestors}}^{t_{\text{sum}}}E^{tH} \right) \begin{bmatrix} \tilde{B}^{t1}E^{t1} \\ \tilde{B}^{t2}E^{t2} \end{bmatrix}^{H}$$

- Step2:

$$G_{2,\text{proj}}^{t} \underset{O(k^3)}{\overset{\text{SVD}}{=}} PDP^{H}$$

- Step3:

$$\begin{bmatrix} \tilde{E}^{t1} \\ \tilde{E}^{t2} \end{bmatrix} = P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \qquad \tilde{B}^{t} = \begin{bmatrix} \tilde{E}^{t1} \\ \tilde{E}^{t2} \end{bmatrix}^{H} \begin{bmatrix} \tilde{B}^{t1}E^{t1} \\ \tilde{B}^{t2}E^{t2} \end{bmatrix}$$

Case 3: Large Scale Array of On-chip Buses

Buses: 4x4 to 64x64

Dimensions: 2μm x 2μm x 20μm

Frequency point: 10 GHz

Number of Current Unknowns:
3968 to 1,015,808

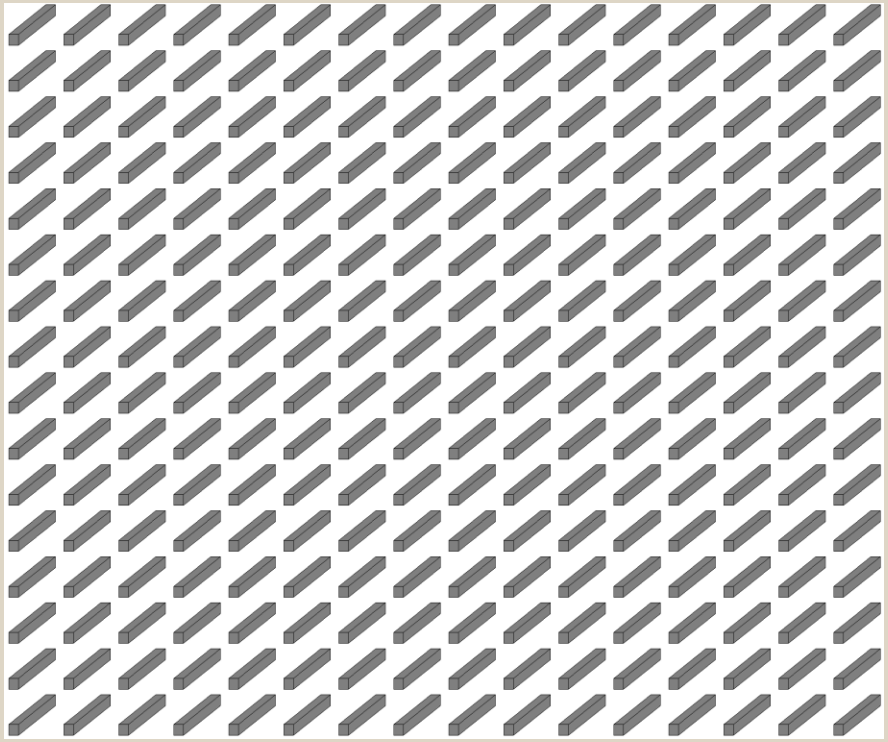
Total number of Unknowns:
6528 to 1,671,168

For 10^{-4} Accuracy

Average *Interpolation* Rank: 34

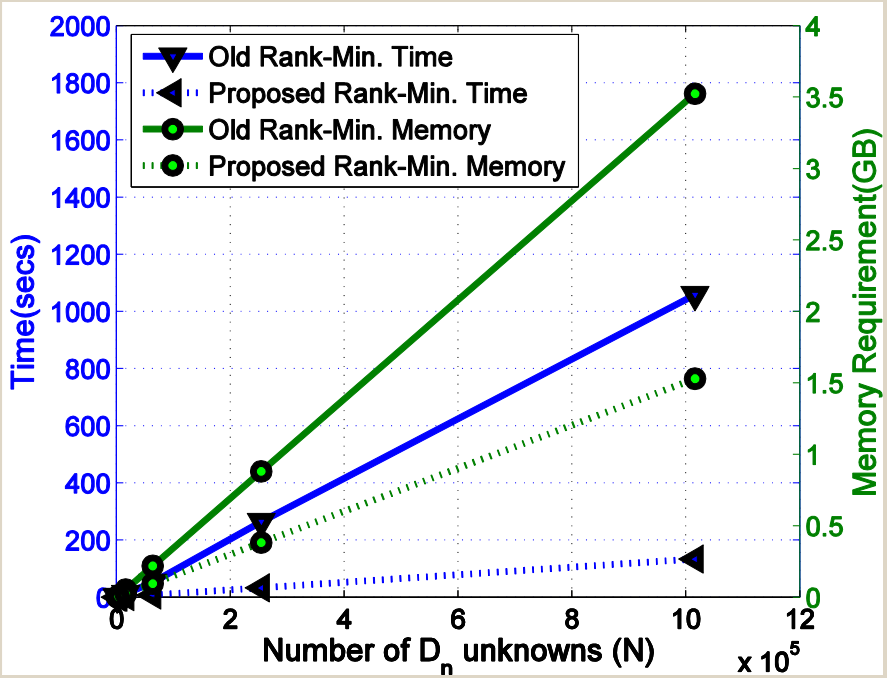
Rank after *rank-minimization*: 11

Neighbor Separation : 10 μm



A 16 x 16 On-Chip Bus Array

Comparison Of Rank-Minimization Procedures

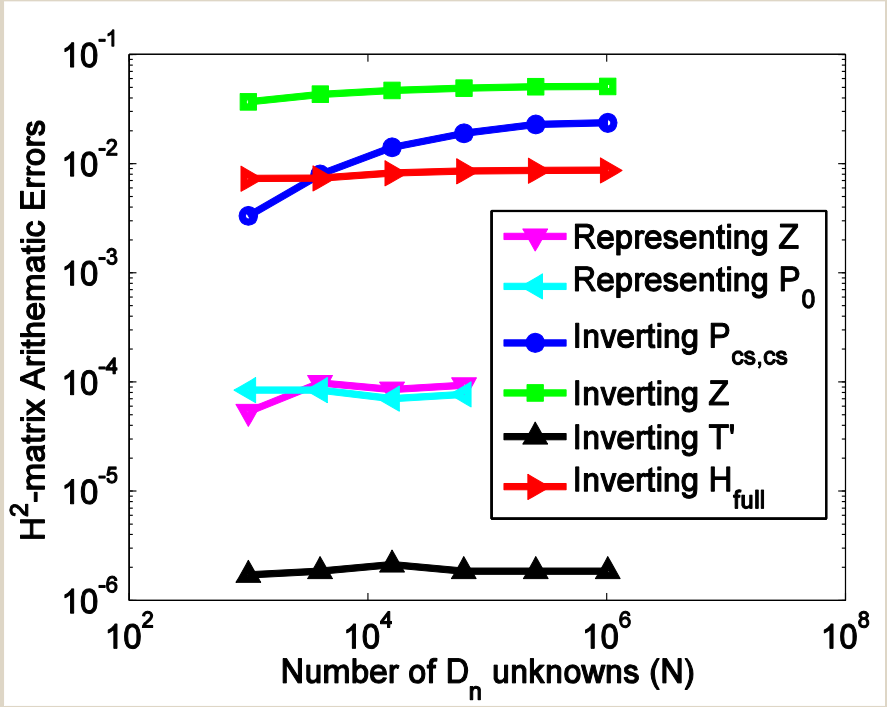


- Interpolation Rank, k^i_{avg} = 34
- Old Scheme Min. Rank , k'_{avg} = 31
- New Scheme Min. Rank, k_{avg} = 11

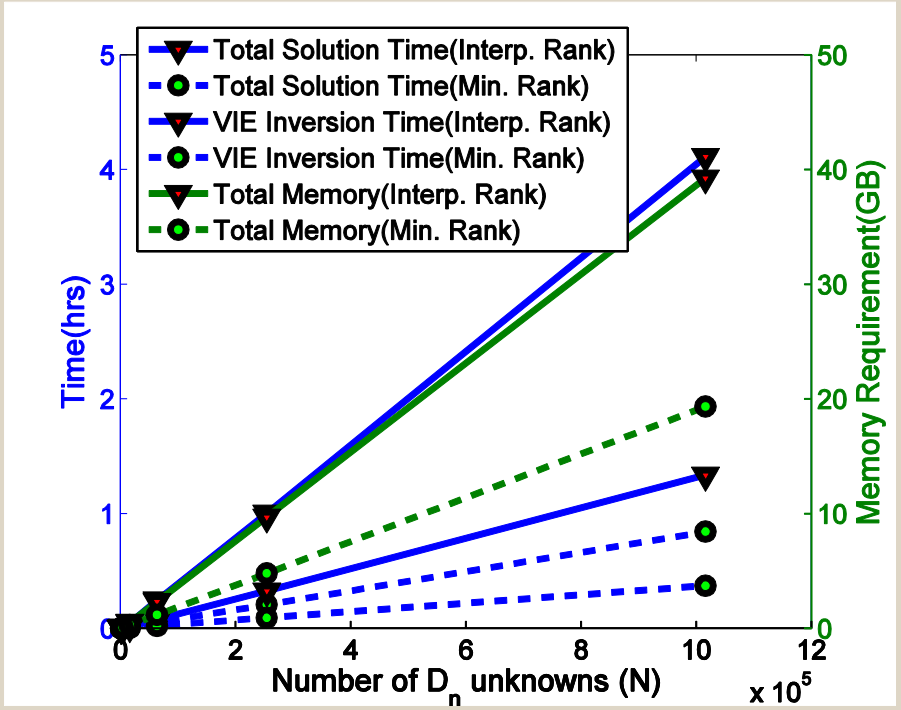
For Admissible Blocks

- Memory Savings ~ 9 times
- Inverse Time Savings ~ 27 times

Case 3: Large Scale Array of On-chip Buses



Error Controllability



Linear Complexity Performance Plot

O(*N*) Iterative & *O*(*NlogN*) Direct VIE Electrodynamic Solvers

Computational Challenges

- $\mathcal{H}^2$-matrices in large-scale Electrodynamic problems
  - Rank in off-diagonal blocks **grows** with electric size
  - For degenerate approximations: **rank, *k* = *O(N)***

$$Memory / MVM = \sum_{l=0}^{L} k^2 2^l = \sum_{l=0}^{L} (N/2^l)^2 2^l = O(N^2 \sum_{l=0}^{L} 2^{-l}) = O(N^2)$$

$$Inversion = \sum_{l=0}^{L} k^3 C_{sp}^2 2^l = \sum_{l=0}^{L} (N/2^l)^3 C_{sp}^2 2^l = O(C_{sp}^2 N^3 \sum_{l=0}^{L} 2^{-2l}) = O(C_{sp}^2 N^3)$$

  - In VIE, theoretically[1]: **rank, *k* = *O(electric size)* = *O*($N^{1/3}$)**

$$Memory / MVM = \sum_{l=0}^{L} (N/2^l)^{2/3} 2^l = O(N^{2/3} \sum_{l=0}^{L} 2^{l/3}) = O(N^{2/3} N^{1/3}) = \boxed{O(N)}$$

$$Inversion\ Cost = \sum_{l=0}^{L} (N/2^l)^{3/3} C_{sp}^2 2^l = O(C_{sp}^2 N \sum_{l=0}^{L} 1) = \boxed{O(C_{sp}^2 N \log N)}$$

  - SVD-rank is attainable by rank-minimization procedure, but…

[1] W.Chai and D. Jiao, " A theoretical study on the rank of integral operators for broadband electromagnetic modeling from static to electrodynamic frequencies," *IEEE Trans. on Components, Packaging and Manufacturing Tech.*, Dec. 2013

Computational Challenges

- Pre-processing cost becomes *the* bottle-neck
  - For degenerate approximation techniques:
    - $O(N^2)$ memory for initial representation
    - $O(N^3)$ operations for rank-minimization
    - Practical only for medium sized analysis
    - Large-Scale analysis---not practically feasible
  - For non-degenerate approximation techniques:
    - Optimal memory cost for initial representation
    - Optimal operation cost for rank-minimization
    - Making possible large-scale analysis
    - e.g. (ACA + r-SVD) methods for $\mathcal{H}$-matrix construction

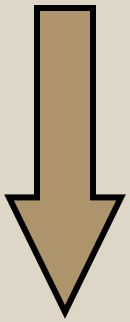
Proposed Large-Scale Electrodynamic Analysis

- **Stage1: ACA and r-SVD based initial construction**

For Each Admissible Pair (t,s)

$$G^{(t,s)} = A_{\#t\times k'} B^{T}_{\#s\times k'}$$

Adaptive-Cross-Approximation



$$r-SVD:$$
$$Step-1: A = Q_A R_A \quad ; \quad B = Q_B R_B$$
$$Step-2: R_A R_B^{T} \overset{SVD}{\rightarrow} U\Sigma V^{H}$$
$$Step-3: A = Q_A U\Sigma \quad ; \quad B = Q_B \bar{V}$$

$$G^{(t,s)} = A_{\#t\times k} B^{T}_{\#s\times k} \qquad k \ll k'$$

Proposed Large-Scale Electrodynamic Analysis

- **Stage2: $\mathcal{H}$ to $\mathcal{H}^2$-matrix conversion**

  For Leaf Clusters (t)

  - Step1: $$G_2^t = \sum_{i \in \text{self}} A_i (B_i^T \bar{B}_i) A_i^H + \sum_{i \in \text{parents}} A_i (B_i^T \bar{B}_i) A_i^H$$
  - Step2: $$G_2^t \underset{O(k^3)}{\overset{SVD}{=}} PDP^H$$
  - Step3: $$V^t = P$$

Proposed Large-Scale Electrodynamic Analysis

For Non-Leaf Clusters (t)

- Step1: $G^{t}_{2,proj} = \sum_{i \in self} A_{i_{small}} (B_i^T \bar{B}_i) A^H_{i_{small}} + \sum_{i \in parents} A_{i_{small}} (B_i^T \bar{B}_i) A^H_{i_{small}}$
- Step2: $G^{t}_{2,proj} \underset{O(k^3)}{\overset{SVD}{=}} PDP^H$

$$A^{O(kxk)}_{i_{small}} \equiv \begin{bmatrix} V^{t1} & \\ & V^{t2} \end{bmatrix}^H A_i$$

- Step3: $\begin{bmatrix} E^{t1} \\ E^{t2} \end{bmatrix} = P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$

For Coupling Matrix Generation

$$G^{(t,s)}_{new} = V^t V^{t\,H} A (V^s V^{s\,H} B)^T$$

$$= V^t S^{(t,s)}_{new} V^{sT} \qquad S^{(t,s)}_{new} = (V^{t\,H} A)(B^T \bar{V}^s)$$

Large-Scale Fast VIE Electrodynamic Solvers

- **Stage3: Fast $\mathcal{H}^2$-matrix Solvers**

$$For\ VIE: rank, k = O(electric\ size) = O(N^{1/3})$$

$$Memory / MVM = \sum_{l=0}^{L} (N / 2^l)^{2/3} 2^l = O(N^{2/3} \sum_{l=0}^{L} 2^{l/3}) = O(N^{2/3} N^{1/3}) = \boxed{O(N)}$$

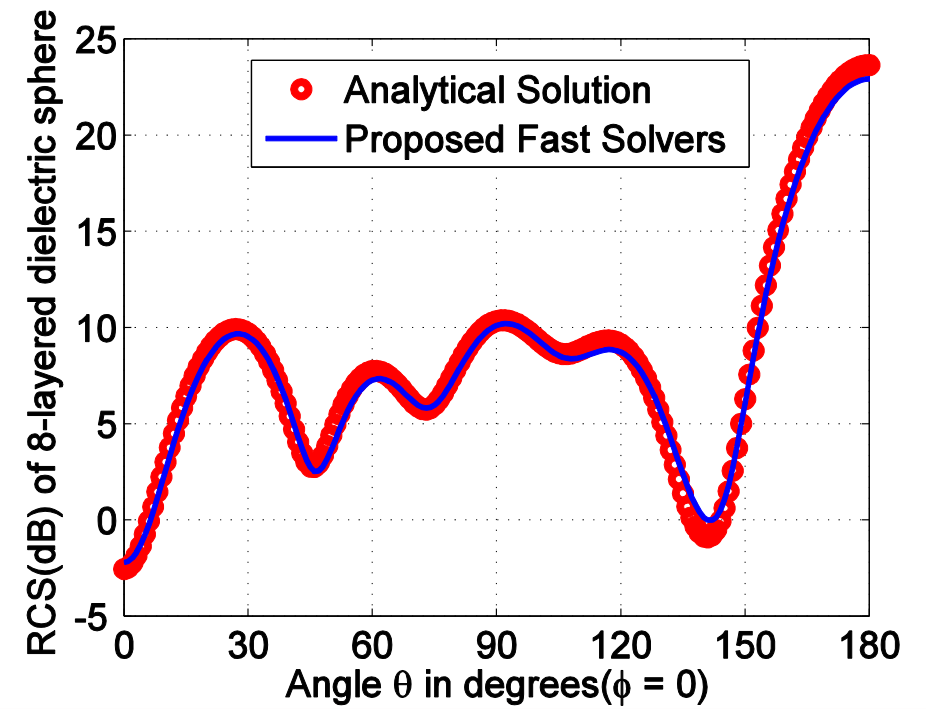
$$Inversion\ Cost = \sum_{l=0}^{L} (N / 2^l)^{3/3} C_{sp}^2 2^l = O(C_{sp}^2 N \sum_{l=0}^{L} 1) = \boxed{O(C_{sp}^2 N \log N)}$$

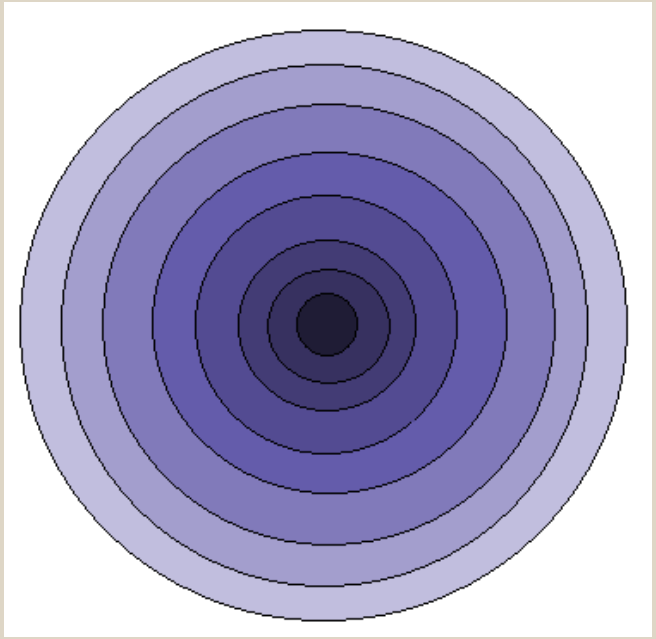
- ***O(N)* VIE Iterative Solver**
- ***O(NlogN)* VIE Direct Inverse Based Solver**
- **Both For Large-Scale Electrodynamic Analysis**

Case 1: Multi-Dielectric Sphere ($k_0a = 6.28$)

Incident field, $E^i = 1.0\, e^{jk_0 z}\, a_x$ Radius of the sphere, a: $1\lambda_0$

Permittivity: [1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0]





Cross-sectional view

Case 2: Large-Scale Dielectric Rod Example

Cross-section: $0.1\lambda_0$ x $0.1\lambda_0$

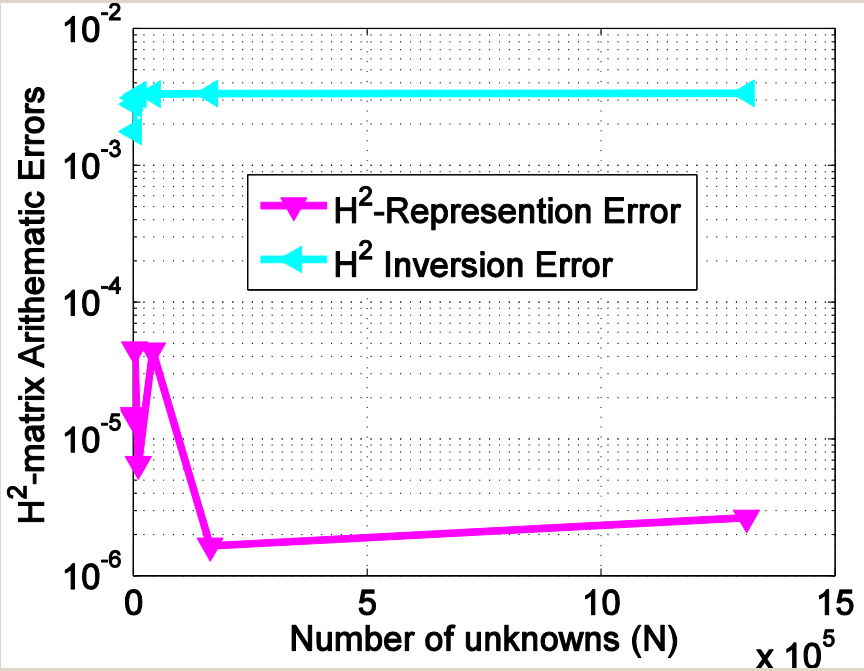
Permittivity: 2.54

Length: $1\lambda_0$ to $8196\lambda_0$

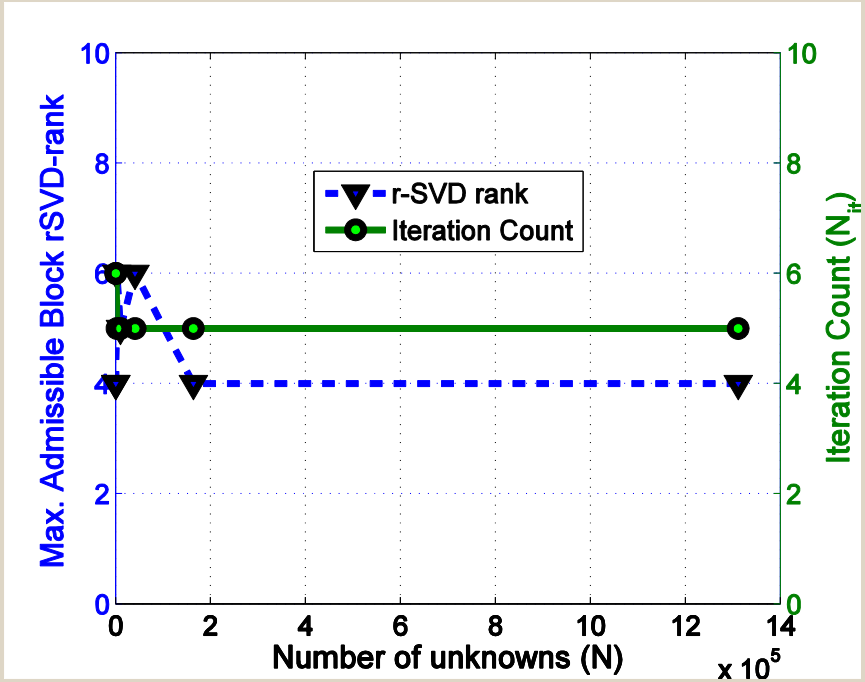
N: 162 to 1,310,736

ACA, SVD truncation $=10^{-4}$

Iteration truncation $=10^{-3}$



Error Controllability



Rank and Iteration Count vs. N

Case 2: Large-Scale Dielectric Rod Example

Cross-section: $0.1\lambda_0$ x $0.1\lambda_0$

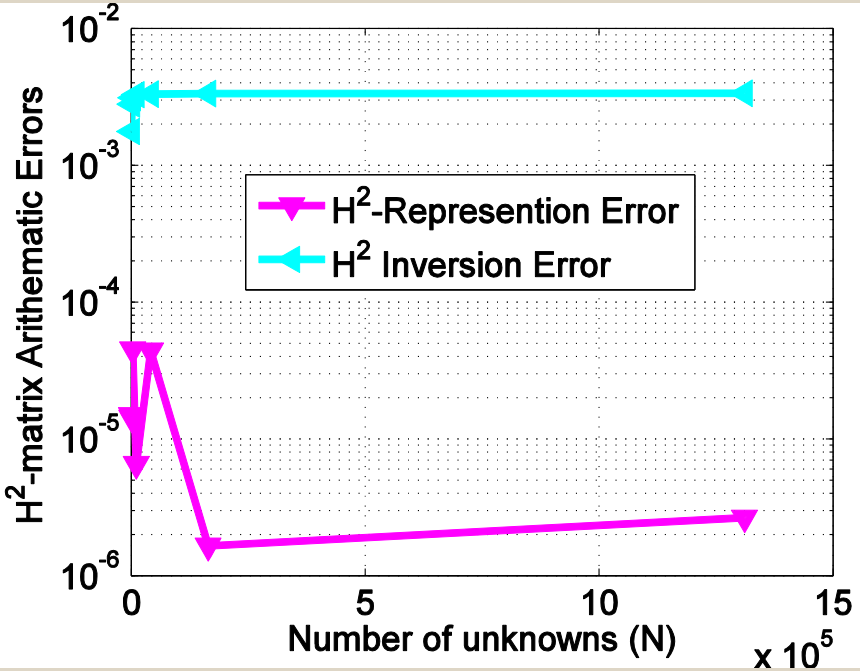
Permittivity: 2.54

Length: $1\lambda_0$ to $8196\lambda_0$

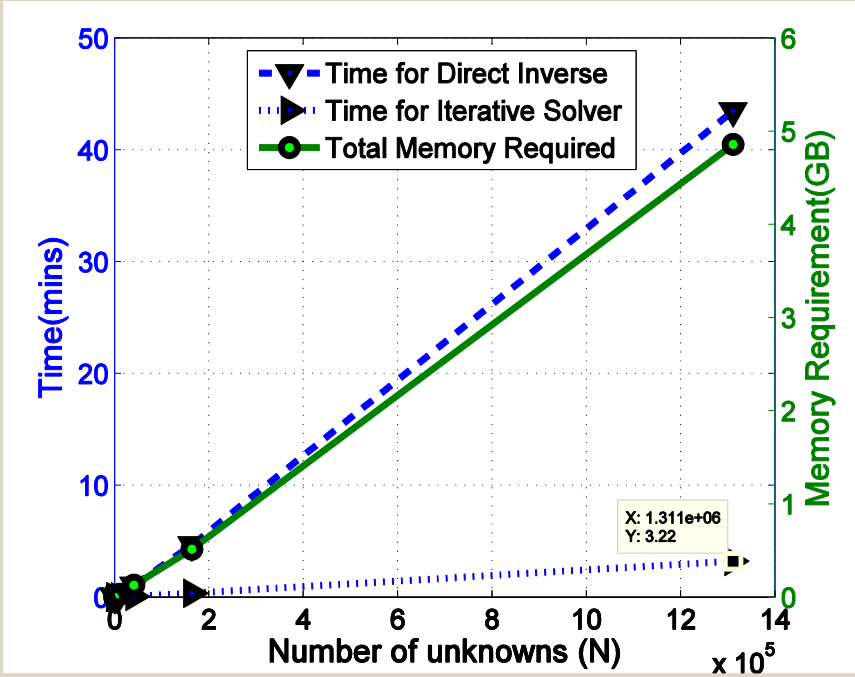
N: 162 to 1,310,736

ACA, SVD truncation =10^{-4}

Iteration truncation =10^{-3}



Error Controllability



Linear Complexity Performance Plot

Case 3: Large-Scale Dielectric Slab Example

Thickness: $0.1\lambda_0$

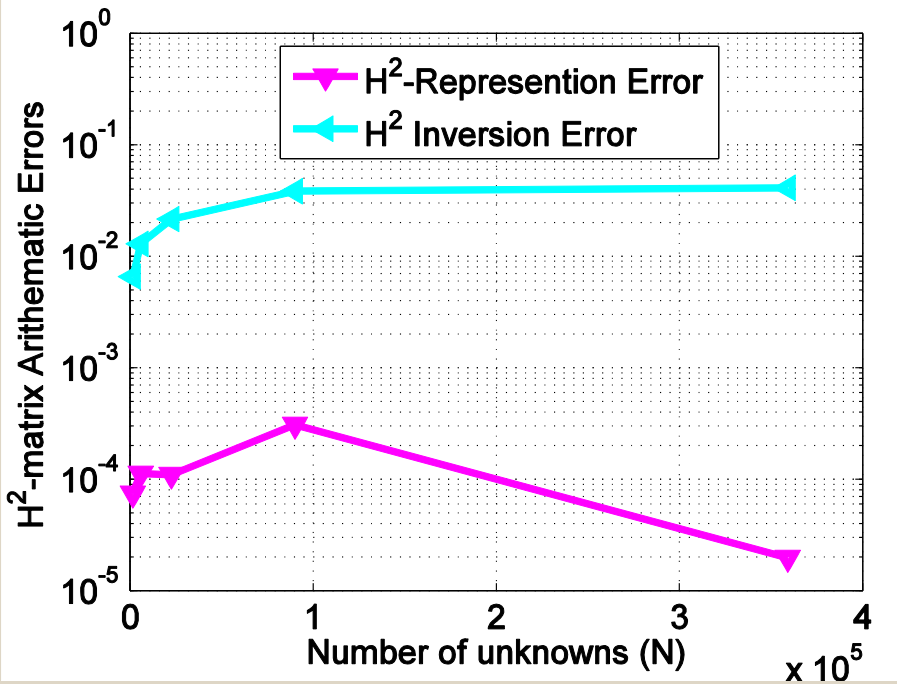
Permittivity: 2.54

Growth: $1\lambda_0 \times 1\lambda_0$ to $16\lambda_0 \times 16\lambda_0$

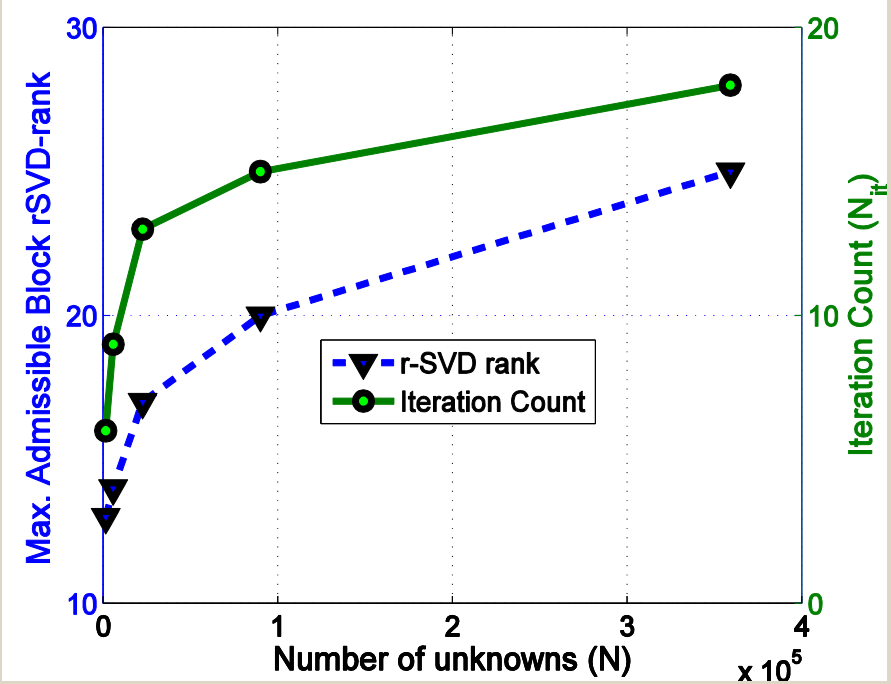
N: 1620 to 359,040

ACA, SVD truncation $=10^{-4}$

Iteration truncation $=10^{-3}$



Error Controllability



Rank and Iteration Count vs. N

Case 3: Large-Scale Dielectric Slab Example

Thickness: $0.1\lambda_0$

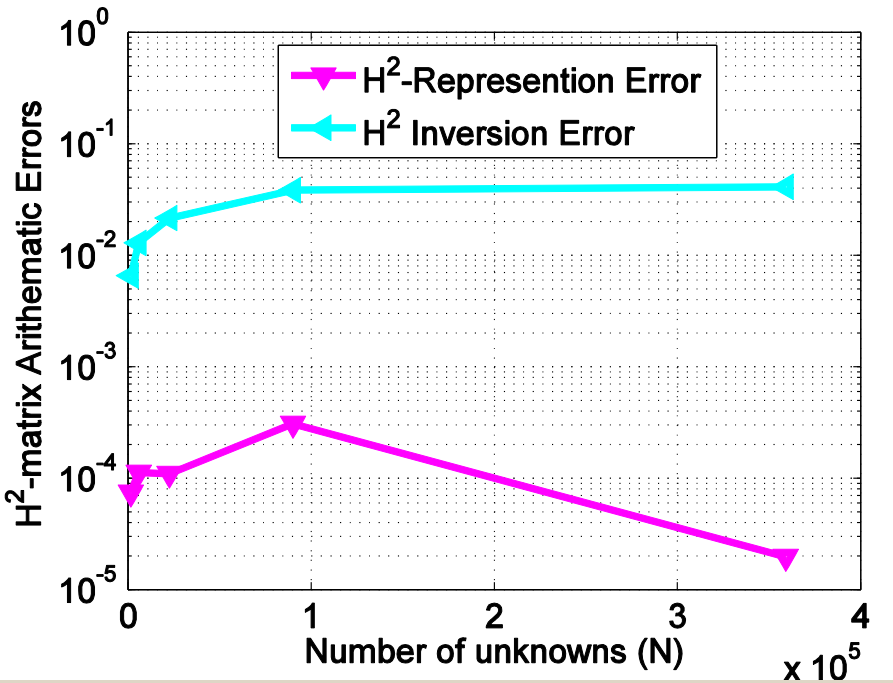
Permittivity: 2.54

Growth: $1\lambda_0 \times 1\lambda_0$ to $16\lambda_0 \times 16\lambda_0$

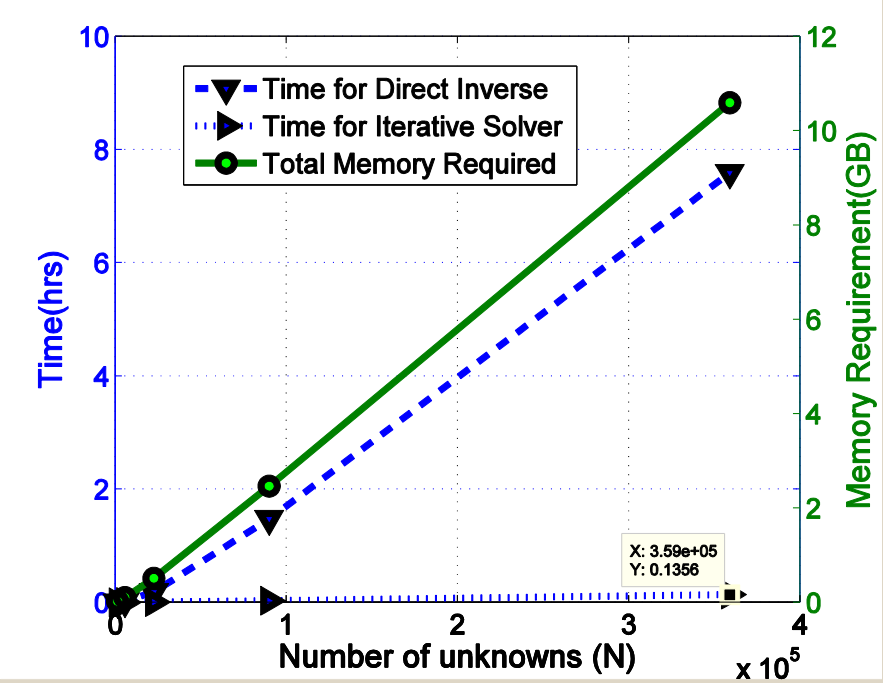
N: 1620 to 359,040

ACA, SVD truncation $=10^{-4}$

Iteration truncation $=10^{-3}$



Error Controllability



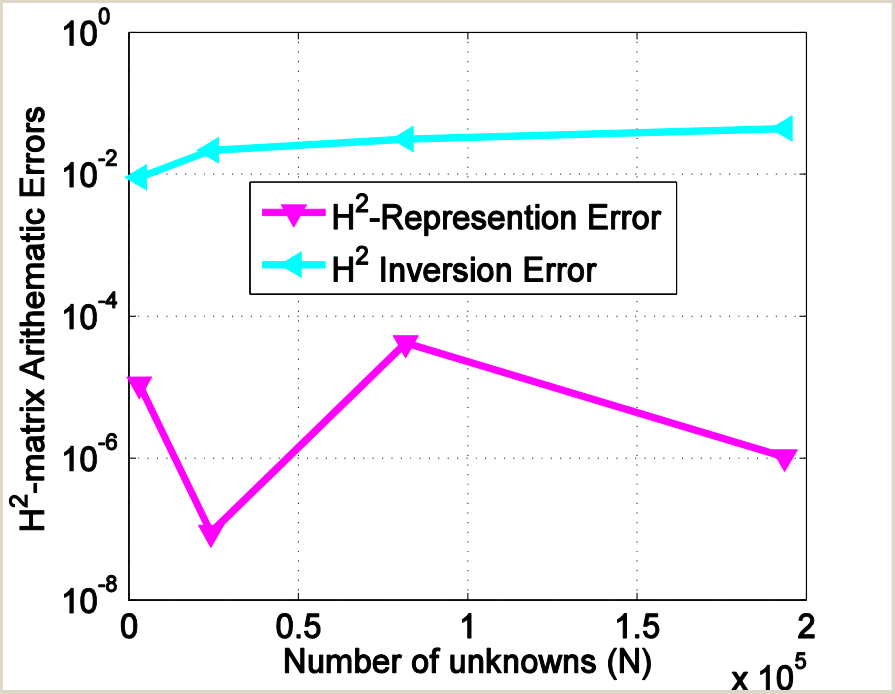
Linear Complexity Performance Plot

Case 4: Large-Scale Array of Dielectric Cubes

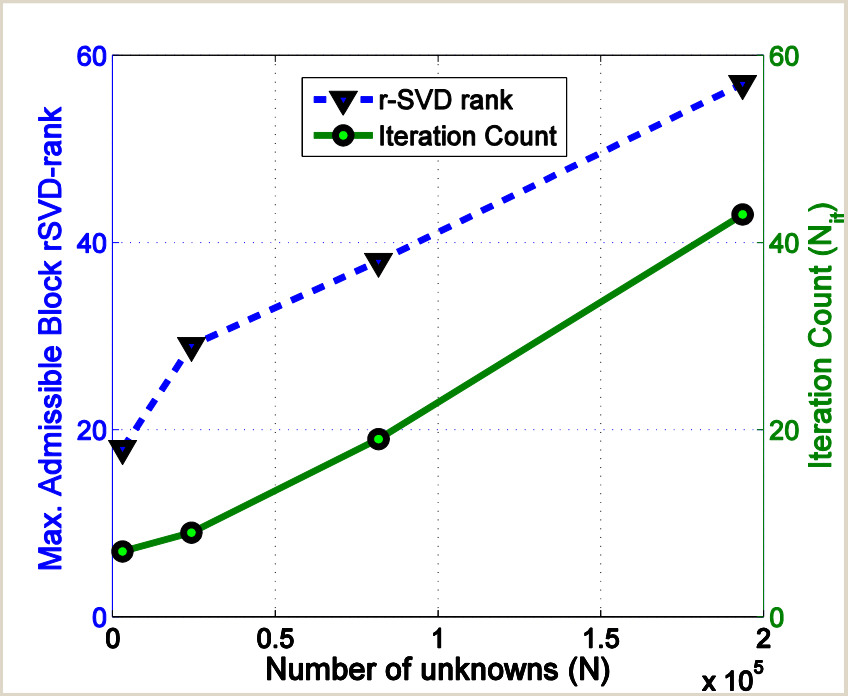
Dimensions : $0.3\lambda_0$ x $0.3\lambda_0$ x $0.3\lambda_0$ Permittivity: 4.0

Array Size: 2 x 2 x 2 to 8 x 8 x 8 N: 3024 to 193,536

ACA, SVD truncation =10^{-4} Iteration truncation =10^{-3}



Error Controllability



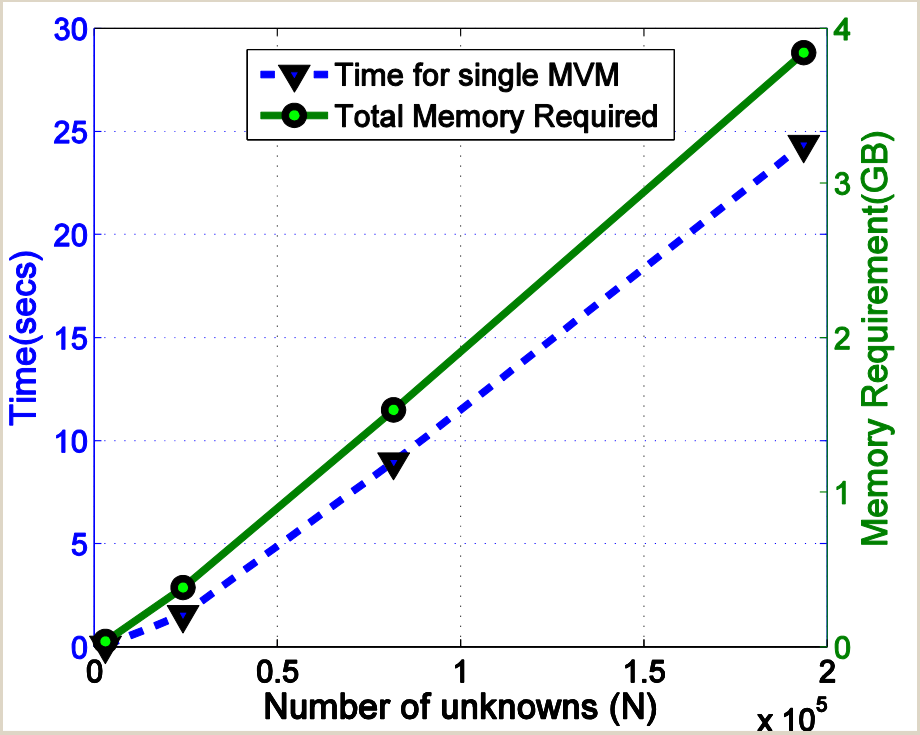
Rank and Iteration Count vs. N

Case 4: Large-Scale Array of Dielectric Cubes

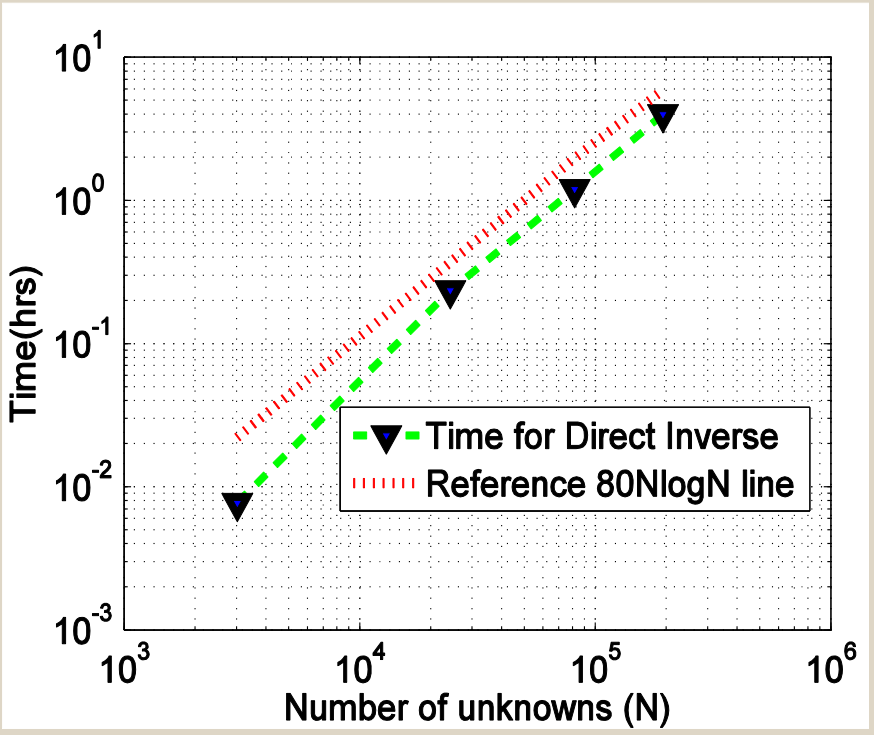
Dimensions : $0.3\lambda_0$ x $0.3\lambda_0$ x $0.3\lambda_0$ Permittivity: 4.0

Array Size: 2 x 2 x 2 to 8 x 8 x 8 N: 3024 to 193,536

ACA, SVD truncation =10^{-4} Iteration truncation =10^{-3}



Linear Complexity Memory / MVM



Inversion Time vs. N

Summary Of Accomplishments

- Novel First-Principles based VIE formulations developed
  - Retain the rigor and the flexibilities of the VIE formulation for wave problems
  - Permit electric-potential based excitation
  - Capable of simultaneous scattering-circuit analysis
- Fast VIE solvers for Large-Scale EM Analysis developed
  - O(N) Direct VIE solver for full-wave circuit analysis
  - O(N) Iterative VIE solver for large-scale electrodynamic analysis
  - O(NlogN) Direct VIE solver for large-scale electrodynamic analysis
- Numerical results validate accuracy and capabilities

Summary Of Accomplishments

Conference Proceedings

- **S. Omar** and D. Jiao, “An H^2-matrix based fast volume integral equation solver for electrodynamic analysis,” *the 27th International Review of Progress in Applied Computational Electromagnetics (ACES)*, March 2011.
- **S. Omar** and D. Jiao, “An explicit inverse based direct volume integral equation solver for electromagnetic analysis,” *the 2011 IEEE International Symposium on Antennas and Propagation*, July 2011.
- J. Zhu, **S. Omar**, W. Chai and D. Jiao, “A rigorous solution to the low-frequency breakdown in the electric field integral equation,” *the 2011 IEEE International Symposium on Antennas and Propagation*, July 2011.
- **S. Omar** and D. Jiao, “An H^2-matrix based fast direct volume integral equation solver for electrodynamic analysis,” *the 28th International Review of Progress in Applied Computational Electromagnetics(ACES)*, April 2012.
- **S. Omar** and D. Jiao, “A novel volume integral formulation for wideband impedance extraction of arbitrarily-shaped 3-D lossy conductors in multiple dielectrics,” *the 2012 IEEE International Symposium on Antennas and Propagation*, July 2012.
- **S. Omar** and D. Jiao, “Solution to the “High” frequency breakdown in EFIE," *the 2013 IEEE International Symposium on Antennas and Propagation*.(**HONORABLE MENTION AWARD**)
- **S. Omar** and D. Jiao, “A new volume integral formulation for full-wave extraction of 3-D circuits in inhomogeneous dielectrics exposed to external fields,” *the 2013 IEEE International Symposium on Antennas and Propagation*.
- **S. Omar** and D. Jiao, “An analytical approach to the low-frequency breakdown of the right hand side and scattered field computation in EFIE,” *the 2013 IEEE International Symposium on Antennas and Propagation*.
- **S. Omar** and D. Jiao, “A new volume integral equation formulation for analyzing 3-D circuits in inhomogeneous dielectrics exposed to external fields,” *the 2013 IEEE International Microwave Symposium*.
- J. Zhu, **S. Omar** and D. Jiao, “The frequency band where the solution to Maxwell's Equations is unknown--a challenge facing the analysis of multiscale problems and its solution.” *IEEE International Symposium on EMC*, 2013.
- **S. Omar** and D. Jiao, “A linear complexity H^2-matrix based direct volume integral solver for broadband 3-D circuit extraction in inhomogeneous materials,” *the 2014 IEEE International Microwave Symposium*.(**BEST STUDENT PAPER--Finalist**)
- **S. Omar** and D. Jiao, “Rank-minimized linear complexity H^2-matrix based direct volume integral equation solver for full-wave 3-D circuit extraction in inhomogeneous materials,” *the 2014 IEEE International Symposium on Antennas and Propagation*. (**BEST STUDENT PAPER--Finalist**)

Summary Of Accomplishments

Peer Reviewed Journal Papers

- **S. Omar** and D. Jiao, "H^2-matrix based fast volume integral equation iterative solver for electrodynamic analysis," *IET Microwaves, Antennas and Propagation*, vol: 7, Iss. 14, pp.: 1145-1153, Nov.2013.(doi: 10.1049/iet-map.2013.0090)
- **S. Omar** and D. Jiao, "A new volume integral formulation for broadband 3-D circuit extraction in inhomogeneous materials with and without external electromagnetic fields," *IEEE Trans. on Microwave Theory & Techniques*, vol: 61, Iss. 12, pp.: 4302-4312, Dec. 2013.(doi: 10.1109/TMTT.2013.2285355)
- J. Zhu, **S. Omar** and D. Jiao, "Solution to the electric field integral equation when it breaks down," in review process, *IEEE Trans. on Antennas and Propagation*, 2014.
- **S. Omar** and D. Jiao, "A linear complexity direct volume integral equation solver for full-wave 3-D circuit extraction in inhomogeneous materials," in review process, *IEEE Trans. on Microwave Theory & Techniques*, 2014.
- **S. Omar** and D. Jiao, "Novel rank-minimized linear complexity H^2-matrix based direct volume integral equation solver for full-wave 3-D circuit extraction in inhomogeneous materials," in preparation.
- **S. Omar** and D. Jiao, "Optimal complexity H^2-matrix based direct and iterative volume integral equation solvers for large-scale electrodynamic analysis," in preparation.

PURDUE
UNIVERSITY

Acknowledgements

- Prof. Dan Jiao

- Committee Members:
  - Prof. Cheng-Kok Koh
  - Prof. Jianlin Xia
  - Prof. Andrew Weiner

- All Lab-mates and Friends

PURDUE
UNIVERSITY

Thank You!